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The effect of a high-frequency progressive vibration on the convective instability of a two-layer fluid☆

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ABSTRACT

The influence of a high-frequency progressive vibration on the onset of thermal convection in a two-layer system of viscous immiscible fluids is investigated. The interface is deformable, the outer walls are rigid, and heat-transfer conditions of a general form are assigned on them. The starting equations are taken in the generalized Oberbeck–Boussinesq approximation. An averaging method is employed. It is shown that the averaged problem contains a vibrogenic external force and vibrogenic stresses that are proportional to the square of the amplitude of the vibration rate. A quasi-equilibrium solution that satisfies the closure condition is found, and its stability is investigated. It is established that, unlike the case of a single-layer fluid, the horizontal component of the vibration influences the onset of convection and have a destabilizing effect. The vertical component stabilizes the two-layer system by increasing the surface tension. The long-wavelength asymptotic is investigated. Calculations are performed for the silicone oil–Fluorinert and acetonitrile–*n*-hexane systems.

There are reviews of the main publications on vibrational convection for regions with a free surface.[1–3] The influence of vertical high-frequency vibration on the onset of convection in a two-layer system has been studied.[4–7] The results of analyses of the averaged equations for a two-layer system with a non-deformable interface, as well as the results of a physical experiment which showed that vertical high-frequency vibrations smooth the interface, have been presented.[4–6] By analogy with a previously proposed approach,[1] an averaging method was employed in the case of vertical vibrations, and model systems that characterize the influence of the physical parameters on the onset of vibrational convection in a two-layer fluid were investigated.[7] The stability of the equilibrium of a two-layer system of fluids with similar densities was investigated; the heat fluxes on the outer boundaries were specified, the interface was deformable, and the thermocapillary effect was neglected; maps of the monotonic and vibrational instability were constructed, and the long-wavelength asymptotic was studied.[8]

The approach previously developed for a single-layer system[1,2] will be used below. An averaging method will be applied to the equations in the generalized Oberbeck–Boussinesq approximation, the equilibrium solution of the averaged problem will be found, and its stability will be investigated.

1. Statement of the problem

Consider a system consisting of two layers of viscous immiscible fluids: an upper layer of thickness H_1 and a lower layer of thickness H_2. The superscript $k=1$ corresponds to values of quantities in the lower layer, and the superscript 2 corresponds to values in the upper layer. Heat-exchange conditions of general form are specified on the rigid outer boundaries so that there is a transverse temperature gradient in each layer. The origin of coordinates is chosen on the flat interface, the x_3 axis is directed along the force of gravity, and $\boldsymbol{\gamma}$ is its unit vector. The interface $x_3=\xi(x_1, x_2, t)$ is assumed to be deformable, and surface tension forces with coefficient $\hat{\sigma}=\sigma_0-\sigma_T\hat{T}^k$ (here $T^1=T^2$) act on it. The temperature is measured relative to its value on the flat equilibrium interface. The fluids are assumed to be slightly non-isothermal, so that the densities depend linearly on the temperature: $\hat{\rho}_k=\hat{\rho}_{0k}(1-\beta_k\hat{T}^k)$. It is assumed that the system as a whole performs vibration along the vector $\mathbf{s}=(\cos\varphi, 0, \sin\varphi)$ according to the law $x_3=(\hat{b}/\hat{\omega})f(\hat{\omega})t$, where f is a 2π-periodic function with a zero mean, $\hat{b}$ is the amplitude of the vibration rate, and $\hat{\omega}$ is the vibration frequency. We write the dimensionless convection equations in the generalized

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Oberbeck–Boussinesq approximation[1,9,10,†]

$$(1-\varepsilon_k T^k)\left(\frac{\partial \mathbf{v}^k}{\partial t}+(\mathbf{v}^k\cdot\nabla)\mathbf{v}^k\right) = -\frac{1}{\rho_{0k}}\nabla p^k+\nu_k\nabla\mathbf{v}^k+$$

$$+(1-\varepsilon_k T^k)(Q_0\boldsymbol{\gamma}-b\omega f''(\omega t)\mathbf{s}),\quad \operatorname{div}\mathbf{v}^k = 0$$

$$\frac{\partial T^k}{\partial t}+(\mathbf{v}^k\cdot\nabla)T^k = \chi_k\Delta T^k$$

$$x_3 = \xi(x_1,x_2,t):\ \mathbf{v}^1 = \mathbf{v}^2,\quad \mathbf{v}^k\cdot\mathbf{l} = \frac{\partial\xi}{\partial t},\quad T^1 = T^2,\quad \kappa_1\frac{\partial T^1}{\partial\mathbf{n}} = \kappa_2\frac{\partial T^2}{\partial\mathbf{n}}$$

$$-(p^1-p^2)n_i+(\tau^1_{ij}-\tau^2_{ij})n_j = -K\sigma n_i-(\nabla_\Gamma\sigma)_i$$

$$x_3 = h_1,-h_2:\ \mathbf{v}^k = 0,\quad B_{1k}\frac{\partial T^k}{\partial x_3}+B_{0k}T^k = b_k \tag{1.1}$$

Here

$$\mathbf{v}^k = (v^k_1,v^k_2,v^k_3),\quad \varepsilon_k = \beta_k A\mathcal{L},\quad \mu_k = \frac{\hat\mu_k\mathcal{T}}{\rho\mathcal{L}^2},\quad \nu_k = \frac{\mu_k}{\rho_{0k}},\quad \chi_k = \frac{\hat\chi_k\mathcal{T}}{\mathcal{L}^2},\quad Q_0 = \frac{g_0\mathcal{T}^2}{\mathcal{L}}$$

$$\sigma = C-MT,\quad C = \frac{\sigma_0\mathcal{T}^2}{\rho\mathcal{L}^3},\quad M = \frac{\sigma_T A\mathcal{T}^2}{\rho\mathcal{L}^2},\quad \nabla_\Gamma(\sigma)_i = \frac{\partial\sigma}{\partial x_i}-\frac{\partial\sigma}{\partial x_j}n_jn_i$$

$$\mathbf{l} = (-\xi_{x_1},-\xi_{x_2},1),\quad \mathbf{n} = (n_1,n_2,n_3) = \frac{\mathbf{l}}{|\mathbf{l}|},\quad \tau^k_{ij} = \mu_k\left(\frac{\partial v^k_i}{\partial x_j}+\frac{\partial v^k_j}{\partial x_i}\right)$$

where v^k are the relative velocities, p^k are the pressures, T^k are the temperatures, K is the average curvature, ε_k are the Boussinesq parameters, b is the dimensionless amplitude, μ_k ν_k, χ_k and κ_k are the dynamic viscosity, kinematic viscosity, thermal diffusivity and thermal conductivity, C is the surface tension coefficient, and M is the Marangoni number. The dimensionless quantities were introduced using the length scale $\mathcal{L}$, the time scale $\mathcal{T}$, the density ρ, the temperature gradient A and the thermal conductivity κ.

2. The high-frequency asymptotic

Next we will consider the case in which the frequency ω is high, the amplitude of the vibration rate b is finite, and the vibration period is less than the characteristic hydrodynamic times, so that the vibrational boundary layers can be disregarded. In addition, we will assume that the condition for using the model of an incompressible fluid holds, i.e., the acoustic wavelength must be much greater than the characteristic dimension.

We will apply the Krylov–Bogolyubov averaging method to problem (1.1) with the assumptions made above. Note that there are also other approaches to the derivation of averaged equations, for example, the multiscale expansion method. However, as was noted,[11,12] all the existing approaches lead to the same averaged equation, although the highest terms in the asymptotic are different. This conclusion is confirmed by the results of numerous studies on vibrational convection (see, for example, Refs 9 and 10).

We will next derive the averaged equations using to the previously developed scheme.[1,2] In addition to the slow time t, we introduce the fast time $\tau=\omega t$. The averaging method gives an asymptotic representation of the solution of system (1.1) in the form of the sum of smooth and fast components with a zero mean over the time τ. In the problem under consideration, we will seek an asymptoteit of the form

$$\mathbf{v}^k = \bar{\mathbf{v}}^k(\mathbf{x},t)+\tilde{\mathbf{v}}^k(\mathbf{x},t,\tau),\quad p^k = \bar p^k(\mathbf{x},t)+\omega\tilde p^k(\mathbf{x},t,\tau)$$

$$T^k = \bar T^k(\mathbf{x},t)+\frac{1}{\omega}\tilde T^k(\mathbf{x},t,\tau),\quad \xi = \bar\xi(x_1,x_2,t)+\frac{1}{\omega}\tilde\xi(x_1,x_2,t,\tau) \tag{2.1}$$

In the scheme of the averaging method, the fast components $\tilde{\mathbf{v}}^k$, $\tilde p^k$, $\tilde\xi$, $\tilde T^k$ are uniquely specified by the conditions, obtained by isolating the leading terms in ω in the equations and boundary conditions of problem (1.1). For them we obtain the problem

$$(1-\varepsilon_k\bar T^k)\frac{\partial\tilde{\mathbf{v}}^k}{\partial\tau} = -\frac{1}{\rho_{0k}}\nabla\tilde p^k+(1-\varepsilon_k\bar T^k)bf''(\tau)\mathbf{s},\quad \operatorname{div}\tilde{\mathbf{v}}^k = 0$$

$$\frac{\partial\tilde T^k}{\partial\tau}+(\tilde{\mathbf{v}}^k,\nabla)\bar T^k = 0 \tag{2.2}$$

[†] See also: Lyubimov DV. Non-linear Problems in the Theory of Rapidly Fluctuating Convective Flows. Doctoral thesis. Perm: Izd Perm Gos Univ; 1994.

$$x_3 = \bar{\xi}(x_1, x_2, t):\ \tilde{\upsilon}_n^1 = \tilde{\upsilon}_n^2,\quad \frac{\partial\tilde{\xi}}{\partial\tau} = -\tilde{\upsilon}_1^k\frac{\partial\bar{\xi}}{\partial x_1} - \tilde{\upsilon}_2^k\frac{\partial\bar{\xi}}{\partial x_2} + \tilde{\upsilon}_3^k,\quad \tilde{p}^1 = \tilde{p}^2$$

$$x_3 = h_1, -h_2:\ \tilde{\upsilon}_3^k = 0 \tag{2.3}$$

As a result, the fast unknowns are expressed in terms of the smooth temperature $\bar{T}^k$.

We will seek a 2π-periodic solution with respect to the fast time τ in the form

$$\tilde{\boldsymbol{\upsilon}}^k = b\mathbf{w}^k(\mathbf{x}, t)f'(\tau),\quad \tilde{T}^k = -b(\mathbf{w}^k(\mathbf{x}, t), \nabla\bar{T}^k(\mathbf{x}, t))f(\tau)$$

$$\tilde{p}^k = b\rho_{0k}\Phi^k(\mathbf{x}, t)f''(\tau),\quad \tilde{\xi} = b(\mathbf{w}^k(x_1, x_2, \bar{\xi}, t), \bar{\mathbf{l}}(x_1, x_2, t))f(\tau) \tag{2.4}$$

Substituting expressions (2.4) into equalities (2.2) and boundary conditions (2.3), for the unknown amplitudes $\mathbf{w}^k$ and Φ^k we obtain the problem

$$(1 - \varepsilon_k\bar{T}^k)(\mathbf{w}^k - \mathbf{s}) = -\nabla\Phi^k,\quad \operatorname{div}\mathbf{w}^k = 0$$

$$x_3 = \bar{\xi}(x_1, x_2, t):\ \rho_{01}\Phi^1 = \rho_{02}\Phi^2,\quad w_n^1 = w_n^2$$

$$x_3 = h_1, -h_2:\ w_3^k = 0$$

Next, we substitute expressions (2.1), taking equalities (2.4) into account, problem (1.1). Averaging over the fast time τ and retaining terms of the order of unity as $\omega \to \infty$, we obtain a closed autonomous system for the smooth components $\bar{\boldsymbol{\upsilon}}^k, \bar{p}^k, \bar{T}^k, \bar{\xi}$

$$(1 - \varepsilon_k\bar{T}^k)\left(\frac{\partial\bar{\boldsymbol{\upsilon}}^k}{\partial t} + (\bar{\boldsymbol{\upsilon}}^k, \nabla)\bar{\boldsymbol{\upsilon}}^k\right) = -\frac{1}{\rho_{0k}}\nabla\bar{p}^k + \nu_k\Delta\bar{\boldsymbol{\upsilon}}^k +$$

$$+ (1 - \varepsilon_k\bar{T}^k)Q_0\boldsymbol{\gamma} + \mathbf{F}_\upsilon^k,\quad \operatorname{div}\bar{\boldsymbol{\upsilon}}^k = 0 \tag{2.5}$$

$$\frac{\partial\bar{T}^k}{\partial t} + (\bar{\boldsymbol{\upsilon}}^k, \nabla)\bar{T}^k = \chi_k\Delta\bar{T}^k \tag{2.6}$$

$$(1 - \varepsilon_k\bar{T}^k)(\mathbf{w}^k - \mathbf{s}) = -\boldsymbol{\nabla}\Phi^k,\quad \operatorname{div}\mathbf{w}^k = 0 \tag{2.7}$$

$$x_3 = \bar{\xi}(x_1, x_2, t):\ \bar{\boldsymbol{\upsilon}}^1 = \bar{\boldsymbol{\upsilon}}^2,\quad w_n^1 = w_n^2,\quad (\bar{\boldsymbol{\upsilon}}^k \cdot \bar{\mathbf{l}}) = \frac{\partial\bar{\xi}}{\partial t}$$

$$(\bar{\tau}_{ij}^1 - \bar{\tau}_{ij}^2)n_j - (\bar{p}_1 - \bar{p}_2)n_i = -(K\sigma + \tau_\upsilon)n_i - (\nabla_\Gamma\sigma)_i \tag{2.8}$$

$$\bar{T}^1 = \bar{T}^2,\quad \kappa_1\frac{\partial\bar{T}^1}{\partial\mathbf{n}} = \kappa_2\frac{\partial\bar{T}^2}{\partial\mathbf{n}},\quad \rho_{01}\Phi^1 = \rho_{02}\Phi^2$$

$$x_3 = h_1, -h_2:\ \bar{\boldsymbol{\upsilon}}^k = 0,\quad w_3^k = 0,\quad B_{1k}\frac{\partial\bar{T}^k}{\partial x_3} + B_{0k}\bar{T}^k = b_k \tag{2.9}$$

As a result of the averaging, vibrogenic forces $\mathbf{F}_\upsilon^k$ appeared in the equations of motion (2.5), and the vibrogenic stresses τ_υ, which are proportional to the vibration parameter $\mathrm{Re}^2 = b^2\langle f^2\rangle$, appeared in the dynamic boundary condition (2.8). These quantities are defined by the expressions

$$\mathbf{F}_\upsilon^k = \mathrm{Re}^2(\mathbf{w}^k, \nabla)\nabla\Phi^k =$$

$$= \mathrm{Re}^2[-\nabla(\mathbf{w}^k)^2/2 + \varepsilon_k(\mathbf{w}^k \wedge \operatorname{rot}\bar{T}^k(\mathbf{w}^k - \mathbf{s}) + (\mathbf{w}^k, \nabla)\bar{T}^k(\mathbf{w}^k - \mathbf{s}))]$$

$$\tau_\upsilon = \mathrm{Re}^2\left(\rho_{01}\frac{\partial\Phi^1}{\partial x_3}\mathbf{w}^1 - \rho_{02}\frac{\partial\Phi^2}{\partial x_3}\mathbf{w}^2, \mathbf{l}\right)$$

We include the potential component isolated in the expression for $\mathbf{F}_\upsilon^k$, assuming that

$$\bar{q}^k = \bar{p}^k - \rho_{0k}Q_0x_3 - \mathrm{Re}^2(\mathbf{w}^k)^2/2$$

Then the remaining terms in this expression are of the order of ε_k or above as $\varepsilon_k \to 0$. Setting $\varepsilon_k = 0$ in the inertial terms, we arrive at the averaged problem

$$\frac{\partial \bar{\mathbf{v}}^k}{\partial t} + (\bar{\mathbf{v}}^k, \nabla)\bar{\mathbf{v}}^k = -\frac{1}{\rho_{0k}}\nabla \bar{q}^k + \nu_k \Delta \bar{\mathbf{v}}^k - \varepsilon_k Q_0 \bar{T}^k \boldsymbol{\gamma} +$$
$$+ \varepsilon_k \mathrm{Re}^2(\mathbf{w}^k \wedge \mathrm{rot}\bar{T}^k(\mathbf{w}^k - \mathbf{s}) + (\mathbf{w}^k, \nabla)\bar{T}^k(\mathbf{w}^k - \mathbf{s}))$$

$$\mathrm{div}\bar{\mathbf{v}}^k = 0, \quad \mathrm{div}\mathbf{w}^k = 0$$

$$\frac{\partial \bar{T}^k}{\partial t} + (\bar{\mathbf{v}}^k, \nabla)\bar{T}^k = \chi_k \Delta \bar{T}^k, \quad (1 - \varepsilon_k \bar{T}^k)(\mathbf{w}^k - \mathbf{s}) = -\nabla \mathbf{\Phi}^k$$

$$x_3 = \bar{\xi}(x_1, x_2, t): \bar{\mathbf{v}}^1 = \bar{\mathbf{v}}^2, \quad w_n^1 = w_n^2, \quad (\bar{\mathbf{v}}^k \cdot \bar{\mathbf{l}}) = \frac{\partial \bar{\xi}}{\partial t}$$

$$(\bar{\tau}_{ij}^1 - \bar{\tau}_{ij}^2)n_j - \left(\bar{q}_1 - \bar{q}_2 - \tau_v + (\rho_{01} - \rho_{02})Q_0\bar{\xi} - \frac{\mathrm{Re}^2}{2}(\rho_{01}(\mathbf{w}^1)^2 - \rho_{02}(\mathbf{w}^2)^2)\right)n_i =$$
$$= -K\bar{\sigma}n_i - (\nabla_\Gamma \bar{\sigma})_i, \quad \bar{T}^1 = \bar{T}^2, \quad \kappa_1 \frac{\partial \bar{T}^1}{\partial \mathbf{n}} = \kappa_2 \frac{\partial \bar{T}^2}{\partial \mathbf{n}}, \quad \rho_{01}\mathbf{\Phi}^1 = \rho_{02}\mathbf{\Phi}^2$$

$$x_3 = h_1, -h_2: \bar{\mathbf{v}}^k = 0, \quad w_3^k = 0, \quad B_{1k}\frac{\partial T^k}{\partial x_3} + B_{0k}\bar{T}^k = b_k \tag{2.10}$$

One of the advantages of the averaging method is that the investigation of the stability of $(2\pi/\omega)$-periodic solutions of the original problem reduces to studying the stability of the corresponding steady-state solutions of the averaged problem. This has been rigorously proved for regions with a rigid boundary.[13,14] In addition, formulae (2.4) enable us to find oscillating corrections and thereby to obtain the leading terms of the high-frequency asymptote (2.1).

3. The equilibrium solution. The eigenvalue problem

We will assume that the heat-transfer conditions are such that problem (2.10) has a quasi-equilibrium solution of the form

$$\mathbf{v}^{0k} = 0, \quad \xi^0 = 0, \quad T^{0k} = A_k x_3 + A_0, \quad \mathbf{w}^{0k} = \left(\cos\varphi \frac{c_k - \varepsilon_k A_k x_3}{1 - \varepsilon_k A_k x_3}, 0, 0\right)$$

$$\mathbf{\Phi}^{0k} = (1 - c_k)\cos\varphi x_1 + \left(x_3 - \varepsilon_k A_k \frac{x_3^2}{2}\right)\sin\varphi$$

$$q^{0k} = -\rho_{0k}\left(\varepsilon_k A_k Q_0 \frac{x_3^2}{2} - \frac{\mathrm{Re}^2}{2}\cos^2\varphi c_k^2\right)$$

$$c_1 = 1 - \frac{\varepsilon_1 \varepsilon_2 A_1 A_2 \rho_{02}(h_1 + h_2)}{\varepsilon_1 A_1 \rho_{01}\ln(1 + \varepsilon_2 A_2 h_2) - \varepsilon_2 A_2 \rho_{02}\ln(1 - \varepsilon_1 A_1 h_1)}, \quad c_2 = \frac{\rho_{01}}{\rho_{02}}(c_1 - 1) + 1 \tag{3.1}$$

which satisfies the closure condition of the pulsation flow rates

$$\int_{-h_2}^{\xi^0} w_1^{02} dx_3 + \int_{\xi^0}^{h_1} w_1^{01} dx_3 = 0 \tag{3.2}$$

Here the gradients A_1 and A_2 are related by the equality $\kappa_1 A_1 = \kappa_2 A_2$.

We linearize system (2.10), setting

$$\bar{\mathbf{v}}^k = \mathbf{v}^{0k} + \bar{\mathbf{u}}^k, \quad \bar{\xi} = \xi^0 + \bar{\eta}, \quad \bar{T}^k = T^{0k} + \bar{\theta}^k$$

$$\mathbf{w}^k = \mathbf{w}^{0k} + \tilde{\mathbf{w}}^k, \quad \mathbf{\Phi}^k = \mathbf{\Phi}^{0k} + \bar{\mathbf{\Phi}}^k, \quad \bar{q}^k = q^{0k} + P^k$$

Assuming that the perturbations are smooth, we introduce stream functions, setting

$$\bar{u}_1^k = \frac{\partial\bar{\psi}^k}{\partial x_3}, \quad \bar{u}_3^k = -\frac{\partial\bar{\psi}^k}{\partial x_1}, \quad \tilde{w}_1^k = \frac{\partial\bar{\zeta}^k}{\partial x_3}, \quad \tilde{w}_3^k = -\frac{\partial\bar{\zeta}^k}{\partial x_1}$$

We next eliminate the pressures P^k and $\bar{\Phi}^k$ and represent the perturbations in the normal form

$$(\bar{\psi}^k(x_1, x_3, t), \bar{\zeta}^k(x_1, x_3, t), \bar{\theta}^k(x_1, x_3, t), \bar{\eta}(x_1, t)) = \\ = e^{\lambda t + i\alpha x_1}(\psi^k(x_3), \zeta^k(x_3), i\alpha\theta^k(x_3), i\alpha\eta)$$

As a result, we arrive at the eigenvalue problem

$$\lambda L\psi^k = \nu_k L^2\psi^k - \varepsilon_k\left[\alpha^2\theta^k Q_0 - \mathrm{Re}^2 A_k\left(\alpha^2\sin\varphi\zeta^k - i\alpha\cos\varphi\frac{c_k - 1}{1 - \varepsilon_k A_k x_3}D\zeta^k\right)\right] - \\ -\varepsilon_k^2\alpha^2\mathrm{Re}^2\cos^2\varphi\frac{A_k(c_k - 1)^2}{(1 - \varepsilon_k A_k x_3)^3}\theta^k \tag{3.3}$$

$$L\zeta^k = \varepsilon_k\left[-\alpha^2\sin\varphi\theta^k + i\alpha\cos\varphi\frac{c_k - 1}{1 - \varepsilon_k A_k x_3}D\theta^k + A_k(x_3 L\zeta^k + D\zeta^k)\right] + \\ +\varepsilon_k^2 i\alpha\cos\varphi\frac{c_k - 1}{(1 - \varepsilon_k A_k x_3)^2}A_k\theta^k \tag{3.4}$$

$$\lambda\theta^k - A_k\psi^k = \chi_k L\theta^k, \quad L = D^2 - \alpha^2, \quad D = d/dx_3 \tag{3.5}$$

When $x_3 = 0$, the boundary conditions have the form

$$\psi^1 = \psi^2, \quad D\psi^1 = D\psi^2, \quad \psi^k = -\lambda\eta \tag{3.6}$$

$$\rho_{01}D\zeta^1 - \rho_{02}D\zeta^2 = -\alpha^2\sin\varphi(\rho_{01} - \rho_{02})\eta + \\ + i\alpha\rho_{01}(c_1 - 1)\cos\varphi(\varepsilon_1\theta^1 - \varepsilon_2\theta^2), \quad \zeta^1 - \zeta^2 = -i\alpha\cos\varphi(c_1 - c_2)\eta \tag{3.7}$$

$$\theta^1 + A_1\eta = \theta^2 + A_2\eta, \quad \kappa_1 D\theta^1 = \kappa_2 D\theta^2 \tag{3.8}$$

$$\mu_1 D^2\psi^1 - \mu_2 D^2\psi^2 + \alpha^2(\mu_1 - \mu_2)\psi^1 = -\alpha^2 M(\theta^k + A_k\eta) \tag{3.9}$$

$$3\alpha^2(\mu_1 - \mu_2)D\psi^1 + \lambda(\rho_{01} - \rho_{02})D\psi^1 - (\mu_1 D^3\psi^1 - \mu_2 D^3\psi^2) + \alpha^2(Q_0(\rho_{01} - \rho_{02}) + C\alpha^2)\eta + \\ + \mathrm{Re}^2(\rho_{01} - \rho_{02})\left[\alpha^2\sin\varphi\zeta^1 - i\alpha D\zeta^2\cos\varphi(c_1 - 1) - \varepsilon_2\alpha^2\cos^2\varphi\frac{\rho_{01}}{\rho_{02}}(c_1 - 1)^2\theta^2\right] = 0 \tag{3.10}$$

On the rigid walls $x_3 = h_1$ and $x_3 = -h_2$, we have the boundary conditions

$$\zeta^k = 0, \quad \psi^k = 0, \quad D\psi^k = 0, \quad B_{1k}D\theta^k + B_{0k}\theta^k = 0 \tag{3.11}$$

Equalities (3.3) contain vibrational terms, beginning from the first order in the Boussinesq parameter ε_k, and boundary conditions (3.7) and (3.10) contain these terms from the zero$^{\text{th}}$ order. If the fluids are homogeneous and have equal densities, gravity and vibration do not influence the stability of the quasi equilibrium.

In order to isolate the leading terms as $\varepsilon_k \to 0$, we seek solutions of Eqs (3.4) in the form

$$\zeta^k = \zeta_0^k + \varepsilon_k\zeta_1^k + \varepsilon_k^2\zeta_2^k + \ldots$$

For the functions ζ_0^k we obtain the problem

$$L\zeta_0^k = 0$$

$$x_3 = 0\colon \zeta_0^1 - \zeta_0^2 = -i\alpha\cos\varphi\eta(\rho_{01} - \rho_{02})S_0, \quad S_0 = \frac{h_1 + h_2}{\rho_{01}h_2 + \rho_{02}h_1}$$

$$\rho_{01}D\zeta_0^1 - \rho_{02}D\zeta_0^2 = -\alpha^2(\rho_{01} - \rho_{02})\sin\varphi\eta$$

$$x_3 = h_1, -h_2\colon \zeta_0^k = 0$$

The solution of this problem consists of the functions

$$\zeta_0^1 = \alpha\eta\Omega(\sin\varphi - i\cos\varphi S_2)(\operatorname{ch}\alpha x_3 - Q_1\operatorname{sh}\alpha x_3) \tag{3.12}$$

$$\zeta_0^2 = \alpha\eta\Omega(\sin\varphi - i\cos\varphi S_1)(\operatorname{ch}\alpha x_3 + Q_2\operatorname{sh}\alpha x_3) \tag{3.13}$$

where

$$\Omega = \frac{\rho_{01}-\rho_{02}}{\rho_{01}Q_1+\rho_{02}Q_2}; \quad Q_k = \operatorname{cth}\alpha h_k, \quad S_k = \rho_{0k}Q_kS_0$$

Retaining only terms that are no higher than the first order in ε_k in Eqs (3.3)–(3.11), we arrive at the eigenvalue problem

$$\lambda L\psi^k = \nu_k L^2\psi^k - \alpha^2\theta^k\mathrm{Gr}_k + A_k\mathrm{Gv}_{1k}(\alpha^2\sin\varphi\zeta_0^k - i\alpha\cos\varphi(c_{0k}-1)D\zeta_0^k]$$

$$\lambda\theta^k - A_k\psi^k = \chi_k L\theta^k \tag{3.14}$$

$$L\zeta_1^k = -\alpha^2\sin\varphi\theta^k + i\alpha\cos\varphi(c_{0k}-1)D\theta^k + A_kD\zeta_0^k \tag{3.15}$$

The boundary conditions for $x_3 = 0$ are

$$\psi^1 = \psi^2, \quad D\psi^1 = D\psi^2, \quad \psi^k = -\lambda\eta \tag{3.16}$$

$$\zeta_1^1 - \varepsilon\zeta_1^2 = i\alpha\cos\varphi(\rho_{01}/\rho_{02}-1)(c_{11}-\varepsilon c_{21})\eta \tag{3.17}$$

$$\rho_{01}D\zeta_1^1 - \rho_{02}\varepsilon D\zeta_1^2 = -i\alpha\rho_{01}\rho_{02}S_0\cos\varphi(\theta^1 - \varepsilon\theta^2) \tag{3.18}$$

$$\theta^1 + A_1\eta = \theta^2 + A_2\eta, \quad \kappa_1 D\theta^1 = \kappa_2 D\theta^2 \tag{3.19}$$

$$\mu_1 D^2\psi^1 - \mu_2 D^2\psi^2 + \alpha^2(\mu_1-\mu_2)\psi^1 = -\alpha^2 M(\theta^k + A_k\eta) \tag{3.20}$$

$$\begin{aligned}&[3\alpha^2(\mu_1-\mu_2) + \lambda(\rho_{01}-\rho_{02})]D\psi^1 - (\mu_1D^3\psi^1 - \mu_2D^3\psi^2) +\\ &+\alpha^2(\rho_{01}-\rho_{02})[Q_0 + \alpha\mathrm{Re}^2\Omega(\sin^2\varphi - \cos^2\varphi S_1S_2)]\eta +\\ &+C\alpha^4\eta + (\rho_{01}-\rho_{02})[\mathrm{Gv}_{1k}\alpha(\alpha\sin\varphi\zeta_1^1 - i\cos\varphi c_{11}D\zeta_0^2) - \varepsilon_2\alpha^2\cos^2\varphi\rho_{01}\rho_{02}S_0^2\theta^2] = 0\end{aligned} \tag{3.21}$$

The boundary conditions for $x_3 = h_1$ and $x_3 = -h_2$ are

$$\zeta_1^k = 0, \quad \psi^k = 0, \quad D\psi^k = 0, \quad B_{1k}D\theta^k + B_{0k}\theta^k = 0 \tag{3.22}$$

where $\mathrm{Gr}_k = \varepsilon_k Q_0$ and $\mathrm{Gv}_{1k} = \varepsilon_k\mathrm{Re}^2$ are the gravitational and vibrational Grashof numbers, $\varepsilon = \varepsilon_2/\varepsilon_1$, and the c_{ik} are the coefficients in the expansion of c_k in series in ε_k:

$$c_1 = c_{01} + \varepsilon_1 c_{11} - \varepsilon_2 c_{21} + \ldots = \frac{\rho_{01}-\rho_{02}}{\rho_{01}+\rho_{02}h_1/h_2} + \varepsilon_1\tilde{A}_1\rho_{02} - \varepsilon_2\tilde{A}_2\rho_{02} + \ldots$$

$$c_2 = c_{02} + \varepsilon_1 c_{12} - \varepsilon_2 c_{22} + \ldots = -\frac{\rho_{01}-\rho_{02}}{\rho_{01}h_2/h_1+\rho_{02}} + \varepsilon_1\tilde{A}_1\rho_{01} - \varepsilon_2\tilde{A}_2\rho_{01} + \ldots$$

$$\tilde{A}_1 = \frac{1}{2}\frac{A_1\rho_{02}(h_1+h_2)}{(\rho_{01}h_2/h_1+\rho_{02})^2}, \quad \tilde{A}_2 = \frac{1}{2}\frac{A_2\rho_{01}(h_1+h_2)}{(\rho_{01}+\rho_{02}h_2/h_1)^2}$$

To investigate the stability of equilibrium (3.1), we will derive the dispersion relation by the standard method, i.e., we solve problem (3.14)–(3.22) without boundary condition (3.20), and then substitute the solution obtained into it. As a result, we arrive at the solvability condition

$$M = \Gamma(\lambda, \alpha, \mathrm{Gr}_k, \mathrm{Gv}_{1k}, \rho_{0k}, \mu_k, h_k, \mathrm{Re}^2, \varepsilon_k, \kappa_k, Q_0, C)$$

which we will not present here because of its length.[‡] Either the eigenvalue parameter λ or the critical values of the parameters can be found from the transcendental equation obtained in explicit form.

[‡] For the detailed mathematical steps, see Zen'kovskaya SM, Novosyadlyi VA. The influence of high-frequency vibration in an arbitrary direction on the onset of convection in a two-layer system with a deformable interface. Article deposited at the All-Union Institute of Scientific and Technical Information (VINITI). 29 June 2007, No. 683-V2007.

In the case of homogeneous fluids, vibrational terms are contained only in boundary condition (3.21). At a fixed wavelength the angle at which vibration does not influence the stability can be found from it:

$$\varphi = \operatorname{arctg}\frac{\sqrt{\rho_1\rho_2 Q_1 Q_2}(1+h_1/h_2)}{\rho_1+\rho_2 h_1/h_2} \tag{3.23}$$

4. The long-wavelength asymptotic

The long-wavelength asymptoteit has been investigated in numerous studies when there is no vibration. The asymptotic of the Marangoni number was constructed in Ref. 15 for a deformable interface in the case of monotonic instability in the form $M = M_0 + \alpha^2 M_1 + \ldots$, where the coefficient M_0 is proportional to the dimensionless gravitation parameter. The long-wavelength asymptotic of the vibrational instability in the form $M = M_0 + \alpha^2 M_1 + \ldots$, $\lambda = \alpha^2\lambda_1 + \ldots$ was considered in Ref. 16, and formulae for M_0 and λ_1 were presented. The leading terms of the asymptotic of the form $M = \alpha^{-2} M_1 + \ldots$, $\lambda = \alpha\lambda_1 + \ldots$ for the case of a non-deformable interface were found in Ref. 17.

In the present study the leading terms in the long-wavelength asymptotic of problem (3.14)–(3.22) were constructed for a deformable interface and homogeneous fluids. In the case of monotonic instability, we found an asymptotic of the form

$$\begin{aligned} M &= M_0 + \alpha^2 M_1 + \ldots, \quad \psi^k = \alpha^2\psi_0^k + \alpha^4\psi_1^k\ldots \\ \theta^k &= \theta_0^k + \alpha^2\theta_1^k + \ldots, \quad \eta = \eta_0 + \alpha^2\eta_1 + \ldots \end{aligned} \tag{4.1}$$

We present the formulae for the leading terms in the case of isothermal boundaries:

$$\begin{aligned} M &= Q_r\frac{2}{3}\frac{h_1h_2(\kappa_1h_2+\kappa_2h_1)(\mu_1h_2+\mu_2h_1)}{(A_1\kappa_1h_2+A_2\kappa_2h_1)(\mu_1h_2^2-\mu_2h_1^2)} + \ldots \\ \psi^1 &= \alpha^2\frac{1}{6}\frac{h_1Q_rh_2\kappa_2x_3(x_3-h_1)^2}{A_1-A_2} + \ldots \\ \psi^2 &= \alpha^2\frac{1}{6}\frac{Q_rh_1^3\kappa_2x_3(x_3+h_2)^2}{h_2(A_1-A_2)} + \ldots \\ \theta^1 &= (x_3-h_1) + \ldots, \quad \theta^2 = \frac{\kappa_1}{\kappa_2}(x_3+h_2) + \ldots \\ \eta &= \frac{\kappa_1h_2+\kappa_2h_1}{\kappa_2(A_1-A_2)} + \ldots \\ Q_r &= Q_0(\rho_{01}-\rho_{02}) - \mathrm{Re}^2(\rho_{01}-\rho_{02})^2\cos^2\varphi\frac{\rho_{01}\rho_{02}(h_1+h_2)^2}{(\rho_{01}h_2+\rho_{02}h_1)^3} \end{aligned} \tag{4.2}$$

The sign of M depends on the ratio $Q_r/(\mu_1h_2^2 - \mu_2h_1^2)$. Under conditions of weightlessness, the value of Q_r is negative when there are vibrations. Note that an asymptotic of form (4.1) does not exist if $\mu_1h_2^2 - \mu_2h_1^2$, as well as in the case of outer boundaries that do not conduct heat.

For vibrational instability we constructed an asymptotic of the form

$$M = \alpha^{-2}M_0 + \ldots, \quad \lambda = \lambda_0 + \ldots$$

It was found that the leading terms do not depend on gravity, surface tension or vibration. The corresponding formulae are not presented here because of there length, but they were used for comparison with numerical results. This asymptotic was previously constructed for a single-layer fluid.[18] In the case where the two-layer system is similar to a single-layer system, the asymptotic values are identical.

5. Results of calculations

The critical Marangoni numbers were calculated for homogeneous fluids in the cases of monotonic and vibrational instability. To test the formulae, the results were compared with the results in Refs 1–3, 15 and 19 and with the asymptotic formulae as $\alpha \to 0$.

We present the results of the calculations for an acetonitrile–*n*-hexane system[19] with a total thickness of 4.5 mm (system *AH*) and a silicone oil–Fluorinert system[20] with an overall thickness of 2 mm (system *SF*) in the case of weightlessness.

The calculations were first performed for small values of the wave number α and were compared with the long-wavelength asymptotic. In the case of monotonic instability, from formulae (4.2) for system *SF* for all values of φ and $\mathrm{Re}^2 = 10$ it was found that

$$\theta_0^1 = -\frac{5}{3}x_3 + 1, \quad \theta_0^2 = -0.87x_3 - 0.35, \quad \eta = -4.29 \tag{5.1}$$

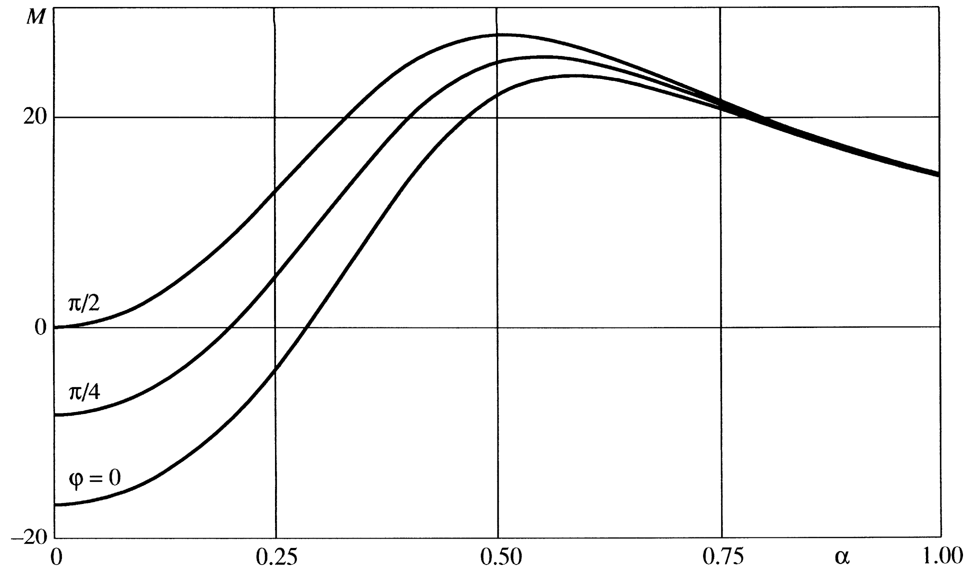

Fig. 1.

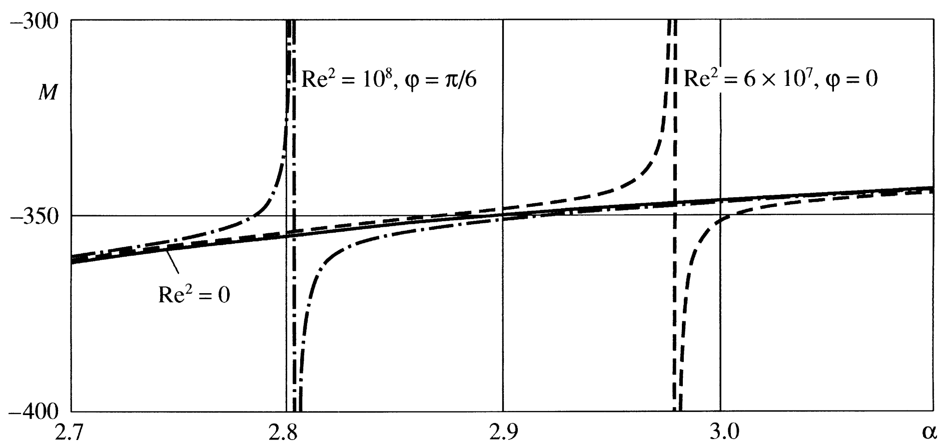

Fig. 2.

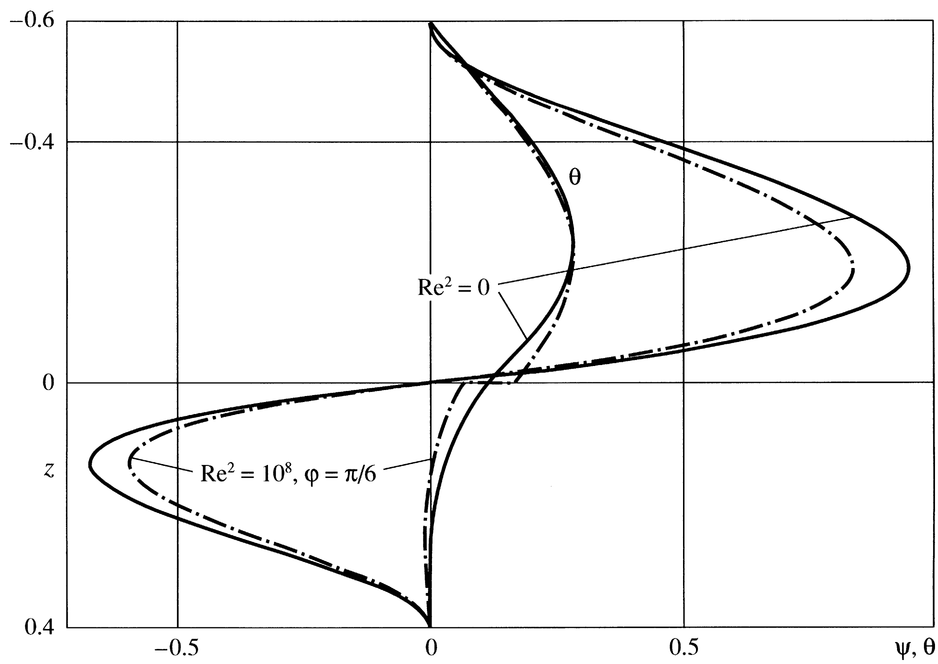

Fig. 3.

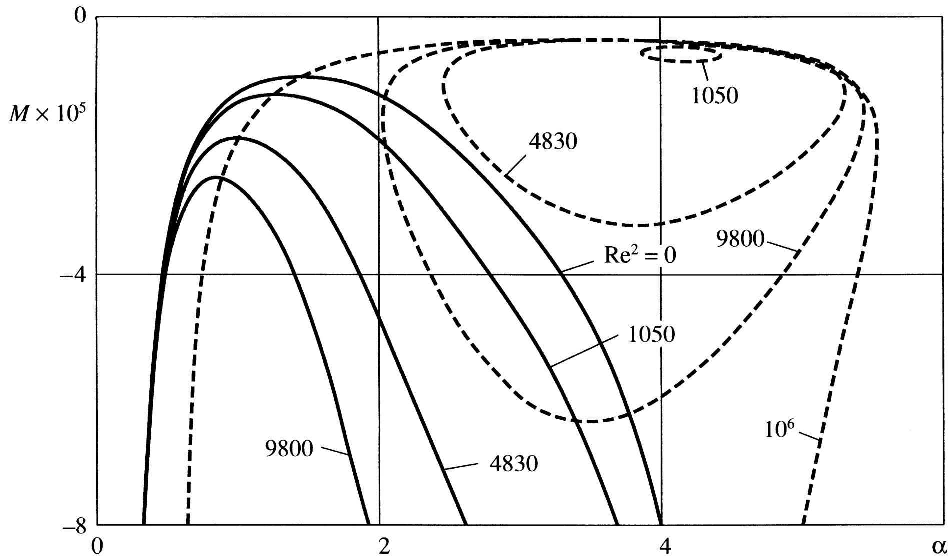

Fig. 4.

The values of M_0 and M_1 and the functions ψ_0^k and ψ_1^k depend on the angle φ:

$$\varphi = \pi/2: \quad M_0 = 0, \quad M_1 = 210.46$$

$$\psi_0^1 = 0, \quad \psi_1^1 = x_3(-50.87 + 169.53x_3 - 141.3x_3^2)$$

$$\psi_0^2 = 0, \quad \psi_1^2 = x_3(-50.87 - 254.35x_3 - 317.93x_3^2)$$

$$\varphi = \pi/4: \quad M_0 = -12.12, \quad M_1 = 195.58$$

$$\psi_0^1 = x_3(2.93 - 9.76x_3 + 8.14x_3^2)$$

$$\psi_0^2 = x_3(2.93 + 14.64x_3 + 18.31x_3^2)$$

$$\varphi = 0: \quad M_0 = -24.24, \quad M_1 = 179.71$$

$$\psi_0^1 = x_3(5.86 - 19.53x_3 + 16.27x_3^2)$$

$$\psi_0^2 = x_3(5.86 + 29.29x_3 + 36.6x_3^2)$$

For the same system *SF* the long-wavelength asymptotic of the vibrational instability gives $M_0 = -84401.735$ and $c_0 = 10.117$. When $\alpha = 0.01$, the results obtained from solving the transcendental equation differ from the asymptotic results by less than 1%. Neutral curves $M(\alpha)$ of the monotonic instability for different values of the angle φ and $Re^2 = 10$, which corresponds to weak vibration, are presented in Fig. 1. As can be seen, the influence of vibration shows up when $0 < \alpha < 1$.

Fig. 2 presents neutral curves $M(\alpha)$ of monotonic instability for system *AH*. The vibration, which includes a horizontal component, leads to the appearance of discontinuities on the monotonic instability curves, which are associated with Kelvin–Helmholtz instability. Fig. 3 presents graphs of the stream function ψ and the temperature θ against the depth of the layer for system *AH* and the value $\alpha = 2.8035$, at which a discontinuity occurs on the curve for $Re^2 = 10^8$ and $\varphi = \pi/6$ in Fig. 2. Normalized values of the stream function and the temperature are plotted along the horizontal axis. It can be seen that the temperature $\theta(z)$ has a discontinuity at $z = 0$, which indicates to deformation of the interface due to vibration.

It was found as a result of the calculations that under the effect of vibration, the neutral vibrational instability curves can consist of two branches, viz., an unclosed branch (the solid curves) and a closed branch (the dashed curves). The branches for $\varphi = \pi/2$ and system *SF* are shown in Fig. 4.

6. Conclusion

We have shown that the influence of high-frequency vibration on a two-layer system differs from its influence on a single-layer system. As was established in Ref. 1, when there is a free deformable boundary, horizontal vibrations ($\varphi = 0$) do not influence the stability of the quasi equilibrium, and when $\varphi \neq 0$, they stabilize it. For a two-layer system the presence of a horizontal component results in destabilization, and the presence of a vertical component results in stabilization. The greatest destabilization is occures when $\varphi = 0$, and the greatest stabilization occures when $\varphi = \pi/2$.

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